

Enlightening complexity: making energy with chaos

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We study the energy harvesting of photons undergoing chaotic dynamics with different complexity degrees. Our theory employs a multiscale analysis, which combines Hamiltonian billiards, time-dependent coupled mode theory and *ab-initio* simulations. In analogy to classical thermodynamics, where the presence of microscopic chaos leads to a single direction for time and entropy, an increased complexity in the motion of photons yields to a monotonic accumulation of energy, which dramatically grows thanks to a constructive mechanism of energy buildup. This result could lead to the realization of novel complexity-driven, energy harvesting architectures.

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Introduction. — One of the most important theoretical dispute of the 19th century concerned the foundations of statistical mechanics and was pointed out by Loschmidt in a famous paradox [1]. He objected, in particular, that it was not possible to deduce an irreversible process from the time-reversible dynamics provided by Boltzmann theory, which was based on the motion of classical particles undergoing elastic collisions [2]. As experimentally ascertained in modern times [3], the physical machinery behind this apparent contradiction is the presence of microscopic chaos, which exhibits an irreversible behavior due to the unpredictability of the resulting dynamics. Sensitivity to the initial conditions relating the thermodynamic system and its environment to the microstate level, in fact, gets exponentially amplified by chaos and culminates into completely irreversible trajectories despite the time reversibility of the microscopic Hamiltonian. The consequence of this result is noteworthy and remarkable, as it establishes a single direction for both time and entropy [4]. This is not only at the basis of our everyday perception of the world, but has also significant implications in several fields of research ranging from quantum mechanics to cosmology [5, 6]. In the language of complex systems, the existence of chaos, or unpredictability, is the signature of a nonvanishing complexity in the dynamics. Complexity stems from a nonzero entropy and may be quantified by the Kolmogorov-Sinai entropy of the corresponding chaotic motion [7]. In light of this interpretation, we may conclude that the influence of a varying complexity even on a simple dynamics, such as the one considered by Boltzmann, may yield dramatic consequences on the resulting macroscopic observables. On the other hand, the classical formulation of statistical mechanics was originally conceived to study ensembles of atoms and molecules interacting through electron wavefunctions, and is therefore characterized by the occurrence of short-range repulsion among the system constituents [8]. With these premises, a natural question

therefore arises and concerns the effects of complexity on particles ensembles showing a bosonic behavior, such as photons, whose nature does not entail any repulsion. Contrary to electrons, photons are energy-carrying particles (of multiples of the light quantum $\hbar\omega$) and their fundamental macroscopic observable is the energy. In this respect, our initial question becomes: *what is the energy harvesting potential of a complex system of photons?* It is not difficult to realize the significant importance of this problem. The theme of energy harvesting, in fact, is one of the leading challenges of contemporary science [9–12], and possesses crucial implications which go beyond applied physics and largely permeate fundamental research on living organisms [13, 14].

In this Letter, we address the above mentioned open question. We begin by considering the simplest integrable, time-reversible Hamiltonian modeling the classical dynamics of photons and then add integrability-breaking terms providing complexity. We then analyze the energy harvesting capacity of the system when photons are injected from an external environment. Although there exist a large literature on the relaxation of energy inside resonators [18], the problem of energy harvesting has never been challenged. The problem is tackled by employing a multiscale analysis. At variance with "simple" systems, in fact, complex dynamics shows the contemporary presence of different scales, each characterized by a specific set of fundamental laws [15]. A single scale highlights just a facet of the phenomenon, and all scales are required to gain a complete understanding over the problem. With reference to the propagation of photons, we identify three main spatio-temporal scales: i) the wave-carrier dynamics, which yields the most detailed description and it is based on the full set of Maxwell's equations; ii) the wave-envelope time-scale, which provides a dynamical description at a larger time with respect to photon's inverse frequency ω^{-1} ; iii) the short-wavelength limit, which constitutes the most ab-

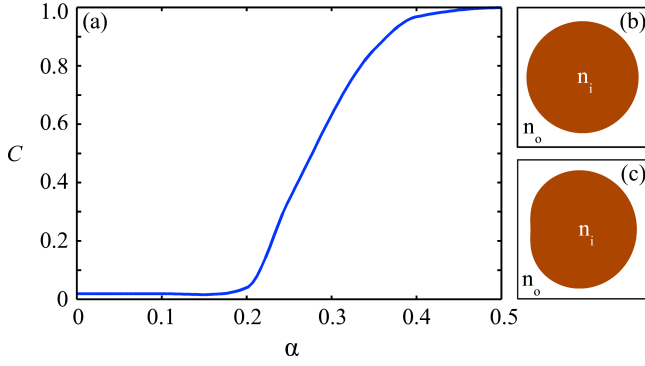


FIG. 1. (Color Online). (a) Complexity \mathcal{C} versus symmetry parameter α ; (b)-(c) open billiard geometry for $\alpha = 0$ (b) and $\alpha = 0.5$ (c).

tract scale and entails a classical description as embodied by Hamilton's equation [16], applied when both time and space are large with respect photon wavelength λ and characteristic time ω^{-1} , respectively. We begin with the short-wavelength limit, where the problem is mathematically defined, and then analyze the dynamics on finer scales up to the most detailed level. The main result of this work is the demonstration of a cooperative dynamics among photons, originating from the complexity of their classical motion and sustaining a constructive mechanism of energy buildup. The efficiency of this process increases with the complexity and, quite remarkably, with the number of photon frequencies injected from the environment, thereby opening the way to the realization of broadband, complexity-driven energy harvesting schemes.

Short-wavelength scale: stochastic transition to complexity. — Stemming from the Maxwell's Hamiltonian modeling photons dynamics at scale i):

$$\mathcal{H} = \frac{1}{2} \int d\mathbf{r}^3 \epsilon(\mathbf{r}) \mathbf{E}^2 + \mu(\mathbf{r}) \mathbf{H}^2, \quad (1)$$

being \mathbf{E} , \mathbf{H} electric and magnetic fields, $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ dielectric and magnetic permittivity, respectively, we apply the WKB transformation $\mathbf{E}, \mathbf{H} \propto e^{iS(\mathbf{r}) - i\omega t}$, thus recasting the original system into the Hamilton-Jacobi (HJ) classical form $d\mathbf{k}/dt = -\partial D/\partial \mathbf{r}$, $d\mathbf{r}/dt = \partial D/\partial \mathbf{k}$, with position \mathbf{x} , momentum $\mathbf{k} = \nabla S$ and $D = D(\omega, \mathbf{k}, \mathbf{r})$ being the dispersion relation of Maxwell's equations. The HJ equations model photons dynamics when both spatial L and time τ scales are large enough, i.e., for $|\mathbf{k}|L \gg 1$ and $\omega\tau \gg 1$. Following the ideas of Boffetta *et al.* [7], we quantitatively define the phase space complexity \mathcal{C} by the Kolmogorov-Sinai (KS) entropy:

$$\mathcal{C} = \int \Lambda(\mathbf{x}, \mathbf{k}) d\nu, \quad (2)$$

being Λ the largest Lyapunov exponent integrated over a measure $d\nu$ of the phase space. However, at variance

with [7] where the standard form of KS entropy is used (i.e., Eq. 2 integrated over the Liouville measure), we define a normalized version of this quantity by employing $d\nu = \Theta(\Lambda)/\Lambda d\mathbf{x}d\mathbf{k}$, with Θ being the Heaviside function. The expression obtained is limited in the range $0 \leq \mathcal{C} \leq 1$. The complexity parameter \mathcal{C} measures the relative volume of the phase space (\mathbf{r}, \mathbf{k}) that encompasses a classical motion with a nonzero entropy. For Hamiltonian phase spaces characterized by regular KAM regions and chaotic seas, Eq. (2) quantifies the growth in size of chaotic components and the increase of the unpredictability of the overall dynamics.

In the case of dielectric uniform media, $D = D_i(\omega, \mathbf{k}) = \omega - c\mathbf{k}/n_i$ with n_i being the index of refraction. In this condition the HJ equations reduce to that of a free particle and the system is integrable. Unpredictability can then be supplemented by the use of a symmetry-breaking potential. In particular, we use a piecewise uniform dielectric geometry by embedding the system into a second uniform material $D_o(\omega, \mathbf{k}) = \omega - c\mathbf{k}/n_o$ characterized by a different refractive index n_o . The region encompassed by n_i defines the harvesting geometry of the problem (i.e., where the energy is stored), while the space delimited by n_o characterizes the environment interacting with n_i . At this spatio-temporal scale, the dynamics of photons injected into the inner region belongs to the class of open billiards [17]: the particle represented by (\mathbf{r}, \mathbf{k}) undergoes a free motion and gets reflected at the boundary \mathcal{S} between n_i and n_o due to the conservation of \mathbf{k} ; energy leaks are then originated from the transmission at \mathcal{S} due to Fresnel's laws [18]. In the billiard dynamics, the complexity of photons is ruled by the symmetry properties of \mathcal{S} . For our problem, we employed a two dimensional polar (ρ, ψ) curve \mathcal{S} , with fixed area V and whose shape originates from the deformation of a circle:

$$\rho = \sqrt{V/\pi - \alpha^2/2} + \alpha \cos \psi, \quad (3)$$

being α a symmetry-breaking parameter (Fig. 1b-c). Figure 1a shows the behavior of the complexity \mathcal{C} when the symmetries of \mathcal{S} are changed through α . As α is increased from the value $\alpha = 0$, we observe a marked stochastic transition at $\alpha \approx 0.2$, with the overall complexity sharply rising due to the breaking of resonant tori. For $\mathcal{C} < 0.5$, the chaotic dynamics is mediated by the survival of KAM islands of integrability; at $\alpha = 0.5$, conversely, all tori break and the system shows the maximum degree of unpredictability, with every input condition leading to chaos. The numerical evaluation of the complexity is carried out by a parallel algorithm, which partitions the billiard phase space $\psi, \theta = \angle \hat{\mathbf{n}}, \mathbf{k}/|\mathbf{k}|$ ($\hat{\mathbf{n}}$ is the normal to \mathcal{S}) into boxes $d\psi d\theta = 10^{-3} \text{rad}^2$, and then numerically evaluates the maximum Lyapunov exponent $\Lambda(\psi, \theta)$. The integration of (2) is then performed with the trapezoidal method.

Wave-envelope limit: photons cooperation effects. — At the wave-envelope level, we model the system dynam-

ics by diagonalizing the Hamiltonian (1) with the aid of time-dependent coupled mode theory [19]. In particular, we decompose the n_i region, which behaves as an open cavity, into a set of modes whose evolution (assuming strong confinement in the cavity) is found to be:

$$\frac{da_k}{dt} = \left[i\omega_k - \left(\frac{1}{\tau_k} \right) \right] a_k + \sqrt{\frac{1}{\tau_e}} c(t), \quad k \in [1, \dots, n], \quad (4)$$

being a_k the amplitude of the k -th cavity mode, with energy $|a_k|^2$, resonant frequency ω_k and total lifetime $\frac{1}{\tau_k} = \frac{1}{\tau_{k0}} + \frac{1}{\tau_e}$, the latter composed by the intrinsic decay rate $1/\tau_{k0}$ and the additional escaping rate $1/\tau_e$ due to the coupling with the external photons source $c(t)$. In order to model a reservoir of photons whose frequencies satisfy $\mathbf{k}L \gg 1$ we considered a broadband source $c(t) = \Theta(t) \int d\omega e^{i\omega t}$, which is switched on at $t = 0$. In this condition, the total electromagnetic energy $\mathcal{H} = \sum_k |a_k|^2$ stored into the cavity is found to be:

$$\mathcal{H} = \int \frac{d\omega}{\tau_e} \sum_k \frac{1 + e^{-\frac{2t}{\tau_k}} - 2 \cos[(\omega_k - \omega)t] e^{-\frac{t}{\tau_k}}}{(\omega_k - \omega)^2 + \frac{1}{\tau_k^2}}. \quad (5)$$

Equation (5) allows to analytically investigate the harvesting capacity of the system when the complexity \mathcal{C} is varied. We begin when the photons dynamics at the short-wavelength scale shows the maximum unpredictability, i.e., for $\mathcal{C} = 1$. In this condition, the long time dynamics $[\psi(t), \theta(t)]$ of photons will be distributed according to the natural measure [17, 20] of the chaotic billiard, regardless of the photons initial position in the phase space. As a consequence, the rate of energy leakage will be independent of $[\psi(0), \theta(0)]$, too. Energy leaks, in fact, relies on the transmission at the boundary \mathcal{S} that, in turn, depends on the time-distribution of $[\psi(t), \theta(t)]$ whose dynamics always converges to the same probability function (i.e., the natural measure). The short-wavelength limit is an abstract scale that does not rely on a particular photons wavelength but equally applies to all wavelengths λ (and frequencies $\omega = 2\pi c/\lambda$) satisfying $\mathbf{k}L \gg 1$. As a result, when considering the more refined wave-envelope time-scale, we can conclude that under the maximum complexity degree the decay rates of the cavity modes should all be independent on the photon's frequency, thus implying $1/\tau_k = 1/\tau$. In addition to that, a second observation can be drawn from the Bohigas-Giannoni-Schmit conjecture [6], which states that the energy spectrum of a generic quantum (or wave) system with underlying classical chaotic dynamics is distributed in the same way as spectra of the standard random matrix ensembles within the same symmetry class. The chaotic billiard dynamics arising from the HJ equations, in particular, belongs to the Gaussian Orthogonal Ensembles (GOE) class and should display repulsion in its wave spectrum, which will be characterized by non overlapping resonant frequencies ω_k . Under this condition, Eq. (5) can be further simplified as the integral

yields significant contribution only for $\omega \approx \omega_k$:

$$\mathcal{H}\tau_e \approx \sum_k \tau^2 (1 - e^{-\frac{t}{\tau}})^2 = \mathcal{N}\tau^2 (1 - e^{-\frac{t}{\tau}})^2, \quad (6)$$

where the sum runs for all the \mathcal{N} resonances ω_k overlapping with the spectrum of $c(t)$. Equation (6), in its remarkable simplicity, predicts that under complex conditions all cavity resonances ω_k cooperate in the accumulation of energy and sustain a coherent buildup process. At the steady state $t \rightarrow \infty$, Eq. (6) yields a form of the thermodynamic equipartition principle:

$$\frac{\partial \mathcal{H}}{\partial \omega} = \text{const} = \frac{\tau^2}{\tau_e}, \quad (7)$$

which predicts a linear energy increase in the "phase-space" given by the spectral domain ω . Conversely, when the photons motion is completely predictable and the system complexity is zero, no natural measure exists for the whole phase space and different input conditions evolve toward different long-time probability distributions. In this regime, each mode resonance exhibits a different characteristic decay rate $1/\tau_k$, and no cooperative dynamics occurs. When the complexity falls outside these two limiting conditions, the phase space will be characterized by both predictable and unpredictable dynamics. In this situation, we expect Eq. (6) to be valid only in chaotic regions where a natural measure can be found, with the net result that only a small subset of mode resonances can contribute to the energy buildup process. The impact of photons cooperation on the process of energy harvesting can be evaluated from the power \mathcal{P} that is transferred into the cavity. From Eqs. (4), this quantity is found to be $\mathcal{P} = \frac{1}{2} \frac{\tau_e/\tau_{k0}}{(1+\tau_e/\tau_{k0})^2}$.

The power \mathcal{P} reaches the maximum when intrinsic and escape decay rates are equal, i.e., for $\tau_{k0} = \tau_e$, and rapidly goes to zero outside this limit. In the absence of any cooperative dynamics between photons, only a small number of resonances ω_k could satisfy this condition, while all the others will inevitably lead to a poor power rate transfer. As a consequence of that, the process energy storage is expected to show a saturation in the efficiency when many frequencies interact with the cavity, due to the limited amount of power that can be transferred under mismatched conditions. In the high complexity case, conversely, thanks to a frequency-independent mechanism of energy buildup, energy harvesting is predicted to grow linearly with the source bandwidth thereby overcoming any bottleneck problem due to a mismatched load.

Wave-carrier limit: ab-initio photons dynamics. —

At the finest scale, we perform *ab-initio* simulations by numerically solving Maxwell's equations with parallel Finite-Difference Time-Domain (FDTD) simulations [21]. *Ab-initio* means "from first principles" and yields the most rigorous approach as it is based on the direct

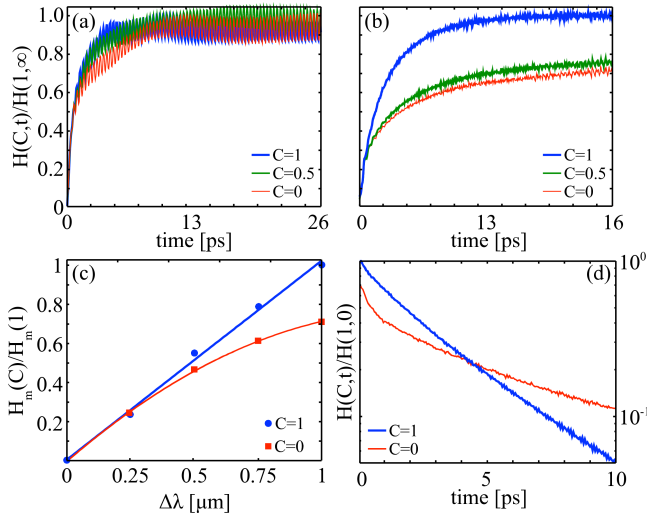


FIG. 2. (Color Online) *Ab-initio* results: (a)-(b) normalized energy \mathcal{H} evolution for (a) single frequency excitation with $\lambda = \lambda_0 = 0.7\mu\text{m}$ and (b) dense frequency comb spanning the range $\Delta\lambda \in [0.2, 1.2]\mu\text{m}$; (c) energy maxima \mathcal{H}_m versus source bandwidth $\Delta\lambda$ (symmetrically increased from λ_0) and complexity \mathcal{C} (marker: FDTD simulation, solid line: numerical fit); (d) log-plot of the energy relaxation dynamics after switching off the source used in (b).

solution of Maxwell's equations without any approximation. Besides that, we also design a possible energy-harvesting device with application in the renewable energy industry. Our device is constituted by a Silicon billiard in air ($n_o = 1$), whose shape follows Eq. (3). At the input, we launch a dense frequency comb with wavelengths λ in the range $\Delta\lambda \in [0.2, 1.2]\mu\text{m}$, thus modeling a broadband source such as a supercontinuum or sunlight, and calculate the electromagnetic energy $\mathcal{H}(\mathcal{C}, t)$ stored into the cavity by a numerical computation of Eq. (1) in the region delimited by n_i (Fig. 2). For a single frequency excitation with $\lambda = \lambda_0 = 0.7\mu\text{m}$, no appreciable distinction is observed as the complexity \mathcal{C} is varied (Fig. 2a). However, as soon as the bandwidth $\Delta\lambda$ of the input source is symmetrically enlarged from the central wavelength λ_0 , we observe a radically different evolution depending on the complexity of the classical motion of photons (Fig. 2b-c). In the complex case $\mathcal{C} = 1$, in particular, photons cooperation settles in and each cavity resonance ω_k coherently contributes to harvest light energy in the system, thus leading to a linear increase of \mathcal{H} . Conversely, when the classical motion of photons is characterized by a zero entropy, saturation occurs because of the mismatch between intrinsic $1/\tau_{k0}$ and escape $1/\tau_e$ decay rates of excited cavity modes (Fig. 2b-c). To further investigate this aspect, we calculate the energy decay in the case of Fig. 2b, i.e., when the input source spans the whole range $\Delta\lambda \in [0.2, 1.2]\mu\text{m}$. This calculation is performed by switching off the frequency comb after $t = 16$ ps, and then monitoring the time evolution

of \mathcal{H} for $\Delta t = 10$ ps (Fig. 2d). In perfect agreement with our predictions, a single exponential decay is observed when $\mathcal{C} = 1$, while different decay times τ_k are manifested for $\mathcal{C} = 0$. The effect of photons cooperation, as seen by comparing Figs 2a-b, is quite dramatic and leads to an energy increase of more than 65%. It is worthwhile remarking that such result is completely nontrivial, as the energy relaxation dynamics for $\mathcal{C} = 1$ is much faster than in the absence of complexity (see Fig. 2d), and it can be predicted only by employing a multi-scale approach that keeps into account all scales of photons dynamics. Besides that, it is not difficult to assess the impact of this result: nearly 50% of the price of any solar module lies in the cost of the Silicon; with such a technology we foresee new harvesting architectures were just a *Si* reshaping (with a constant volume) yield a dramatic increase in the energy stored in the system, with a consequent breakdown of the price to produce a single energy Watt due to a larger energy production.

Conclusions. — We have studied the problem of energy harvesting in complex systems, by employing both analytic theory and *ab-initio* simulations. Time dependent coupled-mode theory and hamiltonian billiard dynamics predict the existence of a mechanism of photons cooperation, which is entirely sustained by complexity. This phenomenon, stemming from the existence of a natural measure in the billiard phase-space and verified by first-principle simulations, is observed as a dramatic increase in the electromagnetic energy stored into the system. In complete analogy to classical statistical mechanics where microscopic chaos predicts a single direction for the entropy, unpredictability in photons dynamics yields a single direction for the energy, which grows with the complexity degree of photons classical motion. This result is expected to stimulate further theory and new experiments devoted to the realization of novel, complexity-driven, light harvesting schemes.

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